

Louis Bachelier’s “Theory of Speculation”

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1 Introduction

Louis Bachelier’s 1900 PhD thesis *Théorie de la Spéculation* introduced mathematical finance to the world and also provided a kind of agenda for probability theory and stochastic analysis for the next 65 years or so. The agenda was carried out by succession of the 20th century’s best mathematician and physicists, but the economic side of Bachelier’s work was completely ignored until it was taken up by Paul Samuelson in the 1960s. By that time the mathematics—which certainly was not developed with any view towards applications in economics—was in perfect shape to solve Samuelson’s problems, and quickly led to the Black-Scholes formula, the watershed event in financial economics. The aim of this talk is to give some account of this twin-track development, based on the discussion in the recent book Davis and Etheridge (2006). The text below consists of some abridged extracts from the book.

2 The mathematics

Bachelier gained his mathematics degree at the Sorbonne in 1895. He studied under an impressive lineup of Professors including Paul Appell, Emile Picard, Joseph Boussinesq and Henri Poincaré.

Appell was a prodigious problem solver with little taste for developing general theories and although he gave his name to a sequence of polynomials, his numerous contributions to analysis, geometry and mechanics are little remembered today. Picard’s name, by contrast, is familiar to any undergraduate mathematician. It is attached to theorems in analysis, function theory, differential equations and analytic geometry. He also had the reputation for being an excellent teacher. In his obituary of Picard in 1943, Hadamard wrote ‘A striking feature of Picard’s scientific personality was the perfection of his teaching, one of the most marvellous, if not the most marvellous, that I have ever known’. Boussinesq made contributions across mathematical physics, notably in the understanding of turbulence and the hydrodynamic boundary layer. It is from Boussinesq that Bachelier learned the theory of heat and it was on Boussinesq’s work in fluid mechanics that Bachelier’s second ‘thesis’ (really an oral examination) was based. The purpose of the second thesis was to test the breadth and teaching abilities of the candidate. Bachelier’s subject involved the motion of a sphere in a liquid and had the less than catchy title ‘*Résistance d’une masse liquide indéfinie pourvue de frottements intérieurs, régis par les formules de Navier, aux petits mouvements variés de translation d’une sphère solide, immergée dans cette masse et adhérente à la couche fluide qui la touche*’.

His first degree required Bachelier to pass examinations in mechanics, differential and integral calculus and astronomy. In 1897, he also took Poincaré’s exam in mathematical physics.¹

¹See Taquu (2001).

Poincaré was Professor of Mathematical Physics and Probability at the Sorbonne during Bachelier's first degree, transferring to the Chair of Celestial Mechanics in 1896. He introduced Bachelier to probability theory, a subject which had rather fallen out of fashion in France since the great treatise of Laplace *Théorie analytique des probabilités* whose final edition was published in 1820. But nonetheless there was a considerable body of work to draw upon, generally couched in the language of gambling, and Bachelier had Joseph Bertrand's highly accessible book *Calcul des Probabilités* from which to learn the basics.

This training in probability and the theory of heat, combined with hands-on knowledge of the stock exchange, provided Bachelier with the tools that he needed to write his remarkable thesis. Precisely this combination of mathematical ideas lays the foundation for the modern theory of *Brownian motion*. Bachelier is generally now credited with being the first to introduce this mathematical process, but there are earlier claims. In 1880, Thorvald Thiele (who taught Neils Bohr mathematics as Professor of Astronomy in Copenhagen) published (simultaneously in Danish and French) an article on time series which effectively creates a model of Brownian motion.

Bachelier's Brownian motion arises as a model of the fluctuations in stock prices. He argues that the small fluctuations in price seen over a short time interval should be independent of the current value of the price. Implicitly he also assumes them to be independent of past behaviour of the process and combined with the Central Limit Theorem he deduces that increments of the process are independent and normally distributed. In modern language, he obtains Brownian motion as the diffusion limit (that is as a particular rescaling limit) of random walk.

Having obtained the increments of his price process as independent Gaussian random variables, Bachelier uses the 'lack of memory' property for the price process to write down what we would now call the Chapman-Kolmogorov equation and from this derives (not completely rigorously) the connection with the heat equation. This 'lack of memory property', now known as the Markov property, was formalised by A. A. Markov in 1906 when he initiated the study of systems of random variables 'connected in a chain', processes that we now call Markov chains in his honour. Markov also wrote down the Chapman-Kolmogorov equation for chains but it was another quarter of a century before there was a rigorous treatment of Bachelier's case, in which the process has continuous paths.

The name Brownian motion derives from a very different route. In science it is given to the irregular movement of microscopic particles suspended in a liquid (in honour of the careful observations of the Scottish botanist Robert Brown, published in 1828). It was in Einstein's 'miraculous year', 1905, that he, unaware of Bachelier's work, introduced his mathematical model of Brownian motion, although he is cautious in his claims, saying that 'it is possible that the motions described here are identical with so-called Brownian molecular motion; however, the data available to me on the latter are so imprecise that I could not form a judgement on the question'. Einstein's motivation was quite different from Bachelier's. Inspired by Boltzmann's 1896 and 1898 work on the kinetic theory of matter, he was looking for ways to verify the existence of atoms. Although the ultramicroscope had brought the observation of molecules closer, it remained impossible to measure their velocities. Einstein introduced the mean-square

displacement of the suspended particles as the primary observable quantity in Brownian motion. He proved that under the assumptions of the molecular theory of heat, bodies of diameter of the order of 10^{-3} mm suspended in liquids must perform an observable random motion. His doctoral thesis already contains a derivation of the diffusion coefficient in terms of the radius of the suspended particles and the temperature and viscosity of the liquid. His first paper on Brownian motion, *Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen*, contains a new derivation making use of methods of statistical physics.

Einstein also derives the connection between Brownian motion and the heat equation. First he assumes that each suspended particle executes a motion that is independent of all other particles and the existence of a time interval that is short compared with the times at which observations are made and yet sufficiently long that the motions of any one suspended particle in successive time intervals can be regarded as mutually independent. The argument for the existence of such time intervals is as follows. The physical explanation of Brownian motion is that the suspended particles are subject to haphazard collisions with molecules which lead to impulses. Although by the laws of mechanics, the motion of a particle is determined not just by these impulses, but also by its initial velocity, for a particle of this size, over any ‘ordinary’ interval of time, the initial velocity is negligible compared to the impulses received during that time interval. This means that the displacement of the particle during such a time interval is approximately independent of its entire previous history. Einstein goes on to describe the motion of the suspended particles in terms of a probability distribution that determines the number of particles displaced by a given distance in each time interval. Using the assumed independence property and symmetry of the motion process he shows that this probability distribution is governed by the heat equation. Combined with his expression for the diffusion coefficient, Einstein obtains from this the probability density function and thence an expression for the mean square displacement of a suspended particle as a function of time. He suggests that it could be used for the experimental determination of Avagadro’s number. Bachelier himself believed that his most important achievement was the systematic use of the concept of continuity in probabilistic modelling. He regarded continuous distributions as fundamental objects rather than just mathematical inventions for simplifying work with discrete distributions. From a modern perspective his greatest insight was the importance of *trajectories* of stochastic processes. After defending his thesis, Bachelier published an article in 1901 in which he revises the classical theory of games from what he calls a ‘hyperasymptotic’ point of view. Whereas the asymptotic approach of Laplace deals with the Gaussian limit, Bachelier’s hyperasymptotic approach deals with trajectories and leads to what we would now call a diffusion approximation. Bachelier was well aware of the importance of his work. He wrote in 1924 that his 1912 book *Calcul des Probabilités* (volume 1, volume 2 was never written) was the first that surpassed the great treatise by Laplace.²

Meanwhile, Einstein was quickly informed by colleagues that his predictions really did fit, at least to order of magnitude, known experimental results for Brownian motion and later in

²See Courtault et al. (2000).

1905 he wrote a further paper more boldly entitled *On the theory of Brownian motion*. Here we meet the first discussion of the fact that our mathematical theory predicts that Brownian paths are highly irregular objects. Indeed Einstein points out that his equation for mean square displacement cannot hold for small times t since that equation implies that the mean velocity of the suspended particle over a time interval of length t tends to infinity as $t \rightarrow 0$. This stems from the assumption that the motions of the particle over successive small time intervals are independent, an approximation which breaks down for very small t . He estimates that the instantaneous velocity of the suspended particle changes magnitude and direction in periods of about 10^{-7} seconds.

Experimental results too pointed to the extremely irregular trajectories. In his 1909 article and later in his popular book *Les Atomes*, Perrin describes how the paths apparently have no tangent at any point. He says ‘C’est un cas où il est vraiment naturel de penser à ces fonctions continues sans dérivées que les mathématiciens ont imaginées et que l’on regardait à tout comme des simples curiosités mathématiques, puisque l’expérience peut les suggérer’.

Norbert Wiener liked to quote Perrin’s observations and it was Wiener, in 1923, who finally produced a mathematically rigorous pathwise construction of Brownian motion. His approach was to construct a probability measure on the space of continuous real-valued functions on the positive half-line (in other words on the space of continuous paths in \mathbb{R}) such that the increments in disjoint time intervals are Gaussian. The idea of establishing a mathematical theory of probability based on integration and measure can be traced to Borel who proved a version of the strong law of large numbers in terms of Lebesgue measure in 1909, but in order to discuss the whole time evolution of a stochastic process, rather than just a snapshot of it at some fixed time, one needs a theory of integration and measure on function spaces and this was what Wiener was able to provide.

Wiener’s seminal paper ‘Differential space’ converts his previous work on integration on function space (and consequently integration in infinitely many dimensions) into a study of Brownian motion. The ‘differences’ are the Gaussian increments of the process. He says ‘The present paper owes its inception to a conversation which the author had with Professor Lévy in regard to the relation which the two systems of integration in infinitely many dimensions – that of Lévy and that of the author – bear to one another. For this indebtedness the author wishes to give full credit.’ His paper begins with a justification of Einstein’s model of Brownian motion and cites F. Soddy’s translation of Perrin’s *Brownian motion and molecular reality*: ‘One realises from such examples how near the mathematicians are to the truth in refusing, by a logical instinct, to admit the pretended geometrical demonstrations, which are regarded as experimental evidence for the existence of a tangent at each point of a curve’. Here then is physical substance for his work. The non-differentiability of Brownian paths which he goes on to prove is not just a mathematical curiosity, but reflects Perrin’s observations of highly irregular molecular motion.

The probability measure that he constructed is now known as Wiener measure and the mathematical model of Brownian motion is often called the Wiener process. Having constructed the measure, Wiener verified that for any fixed time t the probability of differentiability in t is

zero, then he establishes that the probability of satisfying a Hölder condition of order $\frac{1}{2} - \epsilon$ over any interval is one. This quantifies the ‘roughness’ of the path. Finally, Wiener gives the law of the Fourier coefficients. From this one can develop (on the interval $(0, 2\pi)$) a function that vanishes at zero in the form

$$X_t = \xi_0 \frac{t}{\sqrt{2\pi}} + \sum_1^{\infty} \frac{\xi_n(1 - \cos nt) + \xi'_n \sin nt}{n\sqrt{\pi}}$$

where $\xi_0, \xi_1, \xi'_1, \dots$ is a sequence of independent $N(0, 1)$ random variables. This is the Fourier-Wiener series implicit in the 1923 work and explicit in his 1933 collaboration with Paley and Zygmund.

Whereas Wiener constructed the continuous process that we call Brownian motion directly, Bachelier’s approach provided a passage from the discrete to the continuous, but Bachelier simply did not have at his disposal the mathematical machinery to make his hyperasymptotic theory rigorous. It was Kolmogorov,³ in the famous 1931 paper *Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung* who made rigorous the passage from discrete to continuous schemes. He does this by extending Lindeberg’s method for proving the Central Limit Theorem to this setting. The Kolmogorov partial differential equations can then be obtained from the difference equations that hold in discrete time. Bachelier’s influence is evident: Kolmogorov credits Bachelier as being the first to make systematic the study of the case where the transition probability $P(t_0, x, t, y)$ depends continuously on time.

Kolmogorov introduced a Markov transition function as a family of stochastic kernels that satisfy the Chapman-Kolmogorov equation. He himself called it the Smoluchowski equation as Smoluchowski had already written down a special case. The central idea of the paper is the introduction of local characteristics at each time t and the construction of transition functions by solving differential equations involving these characteristics. The ‘Bachelier case’ falls into the last part of the paper where he treats a class of real-valued transition functions for which what we now (following Feller) call the ‘drift’ and ‘diffusion’ coefficients can be defined. Under additional regularity assumptions on the transition probabilities he then proves that they satisfy the Fokker-Planck equation.⁴ Kolmogorov’s work had a powerful effect on the development of probability theory. Now the search was on for distribution functions of continuous limiting processes without recourse to passage to the limit from an approximating sequence.

Kolmogorov’s highly influential monograph of 1933, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, transformed the nature of the calculus of probabilities. Although it had already been understood that the basic manipulations of probability were the same as those of measure theory, the relationship between the two was not sufficiently well formulated as to be useful. The principal problem was the definition of conditional probabilities. Bachelier manipulated these in a non-rigorous way with ease. Kolmogorov gave them a firm mathematical foundation by defining conditional probabilities and expectations as random variables whose existence is

³For more on Kolmogorov’s contributions to probability theory we refer to Dynkin (1989) and the other memorial articles in the same volume.

⁴At the time of writing the article, Kolmogorov was unaware of earlier work of Fokker and Planck who had written down this equation in a special form, but after 1934 he referred to it as the Fokker-Planck equation.

guaranteed by the Radon-Nikodým Theorem. Finally ‘calculus of probabilities’ had graduated to a respectable part of mathematics: ‘probability theory’.

The results in Kolmogorov’s monograph lead to a measure on the space of all functions for which the finite-dimensional distributions are specified, but whereas Wiener measure was defined on continuous functions, so that Wiener’s Brownian motion has continuous trajectories, here there is no regularity. The question remained how to restrict the measure to classes of functions with good regularity properties. A first step in this direction was Kolmogorov’s continuity criterion, first published in a joint paper with Aleksandrov in 1936.

At this point in our history it is still the case that paths played little mathematical rôle. For Kolmogorov and Feller alike, the integrodifferential equations that governed the transition probabilities of the processes were the main object of study. But for Bachelier things were different. Although he did not have the mathematical technique at his disposal to rigorously construct Brownian motion as a continuous stochastic process, and many of his results are calculated from asymptotic Bernoulli trial probabilities, it is clear from his thesis and later work that Bachelier did think in terms of trajectories. The most striking example in the thesis, and the one that evidently impressed Poincaré, was the elegant argument that he gives for the reflection principle. The reflection principle is generally attributed to Desiré André who proved it in the purely combinatorial form, as credited by Bachelier. André was a student of Joseph Bertrand and one can actually find the reflection principle in the context of gambling losses in Bertrand’s 1888 book *Calcul des Probabilités*.

Bachelier’s argument justifying the reflection principle is not rigorous because it requires the strong Markov property of Brownian motion. This property was only properly formulated by Doob in the 1940s and was finally established for Brownian motion by Hunt in 1956. Symmetry arguments similar to that proposed by Bachelier for the reflection principle were also used to great effect by Paul Lévy who established many detailed results about the paths of Brownian motion. In the words of Loève in his obituary of Lévy ‘But above all he is a traveller along paths (this is why for him the Markov property is always the strong Markov property)’. And as Doob remarked, ⁵ ‘Lévy was not a formalist and in particular was not sympathetic to the delicate formalism that discriminates between the Markov and strong Markov properties.’

At the same time as Paley-Wiener-Zygmund made explicit the Fourier-Wiener series, Paul Lévy had been thinking along similar lines. Indeed the question of Brownian motion already arises in his 1937 book *Théorie de l’addition des variables aléatoires*. Lévy observed that one can replace the (trigonometric) Fourier basis functions with other choices to obtain other series expansions of the Wiener process

$$X_t(\omega) = \sum a_n(t)\xi_n(\omega).$$

In an elegant paper of 1961, Ciesielski chooses the Haar basis. Then the $a_n(t)$ are triangular functions supported by the dyadic intervals and the study of the series is quite simple. Lévy wrote⁶ ‘At the beginning of 1934, I suddenly noticed that any stable law leads, as does the

⁵See Doob (1970).

⁶Taken from Loève (1973).

Gaussian, to a random function that we can obtain, like that of Wiener, by an interpolation method. I then decided to find the general form of a function $X(t)$ with independent increments, in other words of an additive process...’ The interpolation consists of determining the law of $X(t + \frac{h}{2})$ when $X(t)$ and $X(t + h)$ are known. This led him to the definition of what are now known as ‘Lévy processes’, another class of mathematical models now widely used in financial applications.

Just before the Second World War Bachelier published a new book on Brownian motion and in 1941 a further paper. In his 1970 memoirs Lévy says that he only became aware of this work after the war. Fundamental to Bachelier’s calculations is what he calls the *true* price of a security. All his calculations are in terms of the true price for which ‘the mathematical expectation of the speculator is null’. Combined with the implicit lack of memory property of the price process, Bachelier is saying that the true price is a *martingale*. It was also Lévy who, in 1934, formalised the concept of a martingale (although the name was coined by Ville in 1939). He introduced the concept in an attempt to preserve the law of large numbers while relaxing independence assumptions (similar considerations motivate Markov dependence). Guided by results obtained for sums of independent random variables he initiated the study of martingales, but it was Doob who transformed the subject and through him martingales became a powerful tool in both probability and analysis.

Joseph Doob came to probability theory from complex analysis. His 1932 doctoral thesis was entitled ‘Boundary values of analytic functions’. He saw the connection between martingales and harmonic functions (also published in two short notes by Kakutani in 1944/5) and based on this he worked to develop a probabilistic potential theory. Martingale theory is the focus of one of the chapters (nearly one hundred pages long) of his 1953 book *Stochastic Processes*, one of the most influential books on probability theory ever written. Doob’s work in the area was heavily influenced by Bachelier. In October 2003 he wrote⁷ ‘I started studying probability in 1934, and found references to Bachelier in French texts, along with references to the reflection principle of Desiré André which I think I remember Bachelier used heavily. I looked up both Bachelier and André and learned a lot. Later I learned that Bachelier’s work was rediscovered later by Lévy and others. Of course the rigorous proofs of Bachelier’s results for Brownian motion had to wait for rigorous mathematical definitions of Brownian motion and the development of suitable techniques. As I remember Bachelier’s writing, he scorned other writers and asserted that only he had obtained new results. In his day, and also considerably later, probability was not a respectable part of mathematics and one man’s probability theory was another man’s nonsense. The ideas of Bachelier and André made a permanent impression on me, and influenced my work on gambling systems and later on martingale theory.’

Doob was also one of the first people to study stochastic differential equations. In the introduction to a paper of 1942 he says ‘A stochastic differential equation will be introduced in a rigorous way to give a precise meaning to the Langevin differential equation for the velocity function $\frac{dx(s)}{ds}$.’ But the central figure in the development of stochastic differential equations was the Japanese mathematician Kiyosi Itô. Itô drew on all the great figures in our story so far. Doob

⁷Personal communication to the authors.

had used measure theory to make sense of the intuitive ideas of sample paths. Kolmogorov and Feller had rather emphasized the connections between Markov processes and partial differential equations. Itô's first paper on stochastic integration, written in 1944, is extremely short. There is very little preamble, he refers to Lévy's book and to Doob's 1937 paper, 'Stochastic processes depending on a continuous parameter', for the definition of Brownian motion and to Paley and Wiener's work of 1934 for the special case of the stochastic integral where the integrand is deterministic. However, in his collected works, Itô explains his motivation: "In these papers I saw a powerful analytic method to study the transition probabilities of the process, namely Kolmogorov's parabolic equation and its extension by Feller. But I wanted to study the paths of Markov processes in the same way as Lévy observed differential processes." Thinking about the behaviour of a Markovian particle over infinitesimal time increments, Itô formulated the notion of stochastic differential equation governing the paths of a Markov process. If W is a standard Wiener process, then Itô's equation for the position of a particle following a Markov process could be written as

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t.$$

In his first paper on stochastic integration in 1944 he made sense of the notion of solution to this equation, that is he gave a rigorous mathematical meaning to the stochastic integral. He also states what one might call 'the fundamental theorem of calculus' for functions of Brownian motion. In his second paper of 1951 he stated and proved what is now known as Itô's formula which makes the connection with Kolmogorov's partial differential equations. Whereas the Wiener integral is the integral of a deterministic function against white noise, time playing no rôle, time is crucial in the Itô integral and moreover the integrand can itself be a random function.

In Doob's 1953 book, Itô's stochastic calculus is extended to processes with orthogonal increments and then to processes with conditionally orthogonal increments, that is martingales. However, Doob had to make an assumption. In order to be able to define the stochastic integral with respect to the martingale M , he required the existence of a non-random increasing function $F(t)$ such that $M_t^2 - F(t)$ is also a martingale. For discrete parameter martingales, the analogous property follows from the Doob Decomposition Theorem, which allows one to write a submartingale (uniquely) as the sum of a martingale and a process with increasing paths, started from zero, and with the property that the process at time n is measurable with respect to the sigma-field generated by information available up to time $n - 1$. The continuous time version of this result, under a certain uniform integrability assumption is due to Meyer (1962). Uniqueness of what is now called the *Doob-Meyer decomposition* came a year later in Meyer (1963). In his first paper Meyer proposes, as an application of the decomposition theorem, an extension of Doob's stochastic integral. A systematic development of these ideas is provided by Courrège (1963), but it was left to Kunita and Watanabe (1967) to provide the analogue of Itô's formula for these more general stochastic integrals.⁸

Up to this point, stochastic integration was intimately bound with the theory of Markov

⁸See Jarrow and Protter (2004) for a more thorough account of these developments.

processes. This came about through a measure-theoretic constraint, the underlying filtration of σ -algebras was assumed to be quasi-left continuous. (This means that the process has no *fixed* points of discontinuity.) In 1970, Doléans-Dade and Meyer removed this hypothesis and stochastic integration became purely a martingale theory (or, more precisely, a semimartingale theory). From the mathematical finance point of view, this was a key ingredient in the fundamental papers of Harrison and Kreps (1979) and Harrison and Pliska (1981).

3 The economics

As we have seen, Bachelier and his work appear as a continuous thread through the development of 20th century stochastic analysis. He published his thesis and a book on probability theory before the first world war and was personally known to leading figures in the French probability community throughout his career. His work was cited in two of the century's most influential works on probability, the Kolmogorov (1931) paper on analytic methods in probability and Doob's *Stochastic Processes*. In short, no-one could say they hadn't been told.

The situation in the worlds of economics and finance could not have been more different. Although the 1908 book of de Montessus on probability and its applications devotes a whole chapter to finance which is based on Bachelier's thesis,⁹ his work made almost no impact when it appeared and lay forgotten for more than 50 years before it finally arrived on the desks of financial economists. In the mid-1950s statistician Jimmy Savage sent postcards to his economist friends alerting them to Bachelier's work. The postcard addressed to Paul Samuelson arrived at an opportune moment, because Samuelson was at the time very much concerned with questions of option and warrant valuation and had at least one PhD student – Richard Kruizenga – working in the area. The Bachelier thesis was duly acquired, and Samuelson commissioned the translation by A. James Boness that now appears, together with work on options and price modelling by Kruizenga and by Benoît Mandelbrot and others, in Paul Cootner's 1964 book *The Random Character of Stock Market Prices* (reprinted in 2001). Asked what he learned from Bachelier's thesis, Samuelson said 'it was the tools' – the panoply of mathematical techniques deployed by Bachelier encompassing Brownian motion, martingales, Markov processes, the heat equation. These were just the things needed for Samuelson's own programme.

Would history have been different if economists had been alerted to Bachelier a few decades earlier? We do not believe so. For one thing, there was no great interest in the subject of option valuation. As Cox, Ross and Rubinstein put it in their 1979 paper: 'Options have been traded for centuries, but remained relatively obscure financial instruments until the introduction of a listed options exchange in 1973'. In fact, 'optionality' in various guises is an ubiquitous feature of financial markets, and how to value it is a key component of asset valuation in general, but this point was not widely appreciated before the massive expansion of financial markets activity in the latter third of the 20th century. This expansion, in turn, could not have occurred without the contemporary developments in computer technology. In earlier days there was no way to hedge an option contract: markets were too illiquid, costs too high and information too scanty.

⁹See Taqqu (2001).

Effective management of option risks depends on having a ‘deep’ (implying large) market and trading on a sufficiently fast time scale. Before the modern era of massive computational power and cheap memory, none of this was feasible. Computer technology is the third leg – alongside the economics and the mathematics – on which modern financial markets stand.

An obvious deficiency of Bachelier’s Brownian motion model of asset prices is that the price at any one time, being normally distributed, can be negative.

To remedy this, Samuelson introduced the *geometric Brownian motion* model in which the asset price $S(t)$ is given by

$$S(t) = S(0) \exp(at + \sigma W(t)), \quad (1)$$

where $W(t)$ is Brownian motion and a, σ are constants. Since $W(t) \sim N(0, t)$ we have $\mathbb{E}[e^{\sigma W(t)}] = \exp(\frac{1}{2}\sigma^2 t)$, so if we take

$$a = \alpha - \frac{1}{2}\sigma^2 \quad (2)$$

then $\mathbb{E}[S(t)] = S(0)e^{\alpha t}$. Thus α is the expected growth rate. The parameter σ is known as the *volatility* and measures the standard deviation of log-returns: the standard deviation of $\log(S(t+h)/S(t))$ is $\sigma\sqrt{h}$.

The move from arithmetic to geometric Brownian motion was on one level a simple expedient to secure positive prices, but on another level it had a profound influence on the whole mathematical development of the subject. The exponential is a *non-linear* function, and analysis of nonlinear transformations of Brownian motion necessarily involves Itô calculus. Applying the Itô formula to $S(t)$ given by (1), (2) shows that $S(t)$ satisfies the stochastic differential equation (SDE)

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t). \quad (3)$$

This is a much more intuitive representation than (1): we can immediately see, for example, that the average growth rate is α and that $S(t)$ is a martingale if $\alpha = 0$. And if we want to examine trading strategies for buying and selling the asset then the SDE representation is essential.

As we saw above, by 1960 the Itô integral, SDEs, and the connection with the heat equation were all well understood. But we can hardly say they were well understood by the man or woman in the street, or even in the scientific laboratory. Ordinary ‘Newton-Leibnitz’ calculus is, and has for centuries been, used in sophisticated ways by a huge range of scientific and engineering practitioners, only a small minority of whom would describe themselves as professional mathematicians. In 1960, by contrast, stochastic calculus was a branch of pure mathematics, studied and understood by at most a few hundred specialists around the globe. There were no textbooks taking anything like an applied point of view. This situation did not however last for long, as study of random phenomena became increasingly important in several different applied areas. In engineering, the accent in the space programme era was on dynamical systems, and a major boost to the study of differential equations with random inputs was provided by Rudolf E. Kalman with his introduction of the Kalman filter (Kalman, 1960). In fact there is another connection with our existing cast of characters here, in that both Kolmogorov and (independently) Norbert Wiener had studied linear filtering and prediction problems in connection with military applications in the Second World War. The Kalman filter met with immense success

and continues in widespread use today. Attempts were soon made to extend filtering theory to non-linear systems, a subject that certainly does require the machinery of stochastic calculus. It took a few years, but by about 1966 a correct formulation had been given by Bucy, Kushner, Shiryaev and others in a series of works which even in hindsight look impressively sophisticated.

The net effect of all these developments was that by the late 1960s stochastic calculus was in far better shape from the point of view of end users and several excellent books such as McKean (1969) or, taking a more applied view, Kushner (1967), were available to explain it all in a palatable way.

In economics, once again Samuelson had the inside track in that Henry McKean was just around the corner in the mathematics department at MIT. McKean was in fact working directly with Itô, a collaboration that resulted in their 1965 book *Diffusion Processes and their Sample Paths* and, indirectly, to McKean's later solo production *Stochastic Integrals*. These remain two of the most admired books in the subject. Also in 1965, Samuelson and McKean collaborated over a paper on option pricing (Samuelson, 1965) about which we shall have more to say below. A further recruit to the project was Robert Merton, who arrived as a graduate student at MIT in 1967 with a background in applied mathematics. Merton was probably the first person to appreciate clearly the connection between Itô integrals and trading strategies, and he quickly produced a series of classic papers (collected in his book *Continuous-time Finance*) in which significant problems of financial economics were solved by methods of Itô calculus.

4 En route to Black-Scholes

We will not record in detail the many tortuous steps taken by researchers in search of an option pricing formula prior to the decisive breakthrough by Black and Scholes in 1973. A good impression of the various theories as of the mid-sixties can be gleaned from Cootner's book *The Random Character of Stock Market Prices*. As Harrison and Pliska put it in 1981, 'these theories, developed between 1950 and 1970, all contained *ad hoc* elements, and they left even their creators feeling vaguely dissatisfied'. Our economist friends will probably dislike this analogy, but it seems that the process the financial economists were going through bears some similarity to what was happening across the fence in stochastic analysis at very much the same time. As we have already described, the development of stochastic process theory in the 1960s was closely bound up with Markov processes. One can see this in the original 1966 edition of Meyer's *Probability and Potentials* and in the famous Kunita and Watanabe paper of 1967. As Meyer's *théorie générale des processus* gathered momentum, the relationship with Markov processes gradually withered away, leaving as a final product a *théorie des semimartingales* in which Markov processes played no direct rôle at all.

In a similar manner, the job of the economists in relation to the option pricing problem was to get rid of the economics. It was originally thought, reasonably enough, that since an option is a risky contract its value must have something to do both with other risky assets in the market and with investors' risk preferences. Option valuers were therefore looking at utility functions, economic equilibrium, the Capital Asset Pricing Model and so on. In the context

of ‘complete markets’—the Black-Scholes world—all of these are simply irrelevant. The only ‘economics’ left is the statement that people prefer more to less and a closely related principle, sometimes dignified with the name *The Law of One Price* which, more formally stated, says that two contracts that deliver exactly the same (fixed or random) cash flows in the future must have the same value today. Otherwise one could sell the dearer one, buy the cheaper one, pocket the difference and walk away: an arbitrage opportunity. Of course, a big assumption is buried in this argument, namely ‘frictionless markets’, i.e., the ability to trade long and short in arbitrary amounts with no transaction costs.

Economists will say, correctly, that complete markets constitute a wafer-thin slice of economic activity and it is only on this slice that economic theory is reduced to more-better-than-less. Nonetheless, the impact of the complete markets theory has been overwhelming in terms of the development of capital markets because of its ability to make unambiguous quantitative statements.

Samuelson’s 1965 paper *Rational theory of warrant pricing* is, in the author’s words, ‘a compact report on desultory researches stretching back over more than a decade’. It is one of those papers that contain *ad hoc* elements, but aside from that is notable for three things: the sheer quality of the intuition, the first formal analysis of American options, and the collaboration with McKean, who contributed a lengthy appendix on the American option problem. A *warrant* is an offer by a company to sell shares to investors at a stated price. From our point of view it is the same thing as a call option – there are contractual differences, but they need not detain us. Warrants can generally be exercised at any time up to some fixed expiry time T and are thus American options. There may be no expiry time, so that $T = \infty$, in which case we have a ‘perpetual’ warrant. The empirical facts are that perpetual warrants trade for less than the underlying stock and are exercised over time, depending on movements of the underlying price. The challenge is to determine a ‘rational’ price and the optimal exercise strategy. Samuelson adopted the geometric Brownian motion model (1), so that in particular the expected growth rate is α , i.e. $E[S(t)] = S(0)e^{\alpha t}$. The ‘*ad hoc* assumption’ is that the warrant price has expected growth rate equal to a constant β up to the time when it is optimal to exercise the warrant. Samuelson argues that if $\beta = \alpha$ then it is never optimal to exercise before the expiry date T , so a viable theory requires $\beta > \alpha$, which in any case is necessary to compensate the warrant holder for not receiving dividends before exercise. McKean gives a precise formulation and a complete solution in terms of a free boundary problem in partial differential equations (PDE). In the perpetual case the PDE becomes an ordinary differential equation and the problem can be explicitly solved (otherwise numerical solution is required). The point about all of this from a latter-day perspective is that the solution coincides with Black-Scholes if one interprets β as the riskless rate of interest r , and $\alpha = r - q$ where q is the dividend yield of the stock. In particular the connection with free-boundary problems and the mathematical results slot right into the later theory. It took a surprisingly long time, however, for a rigorous proof to appear showing that the solution of the free-boundary problem provides a unique arbitrage-free price. Some of the background to this is given below.

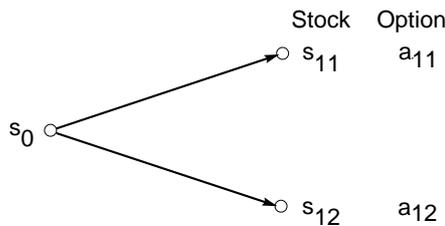


Figure 1: Single-period binomial tree

5 The Binomial Model

To describe the achievements of Black, Scholes and their successors we introduce the reader at this point to the single-period binomial model (originally introduced by Cox, Ross and Rubinstein in 1979). Although at first sight extremely artificial, this model has the big advantage that the whole theory can be described in a couple of pages and the only calculation required is solution of one pair of simultaneous linear equations.

The model is shown in Figure 1. At time 0, an asset has price S_0 equal to some value s_0 . At time 1, its price S_1 is one of two known values s_{11}, s_{12} (we take $s_{11} > s_{12}$). Each of the two values occurs with strictly positive probability, but we *do not* specify what this probability is. The other asset in the market is a riskless bank account paying interest at per-period rate r , so that \$1 deposited at time 0 pays $\$R$ at time 1, where $R = 1 + r$. A contingent claim is written on the asset. This is a contract which is exercised at time 1 and has exercise value $A_1 = \$a_{11}$ if $S_1 = s_{11}$ and $A_1 = \$a_{12}$ if $S_1 = s_{12}$. It could be a call option, so that $A_1 = \max(S_1 - K, 0)$ for some strike price K , but it is not important how the values a_{11}, a_{12} are arrived at; they are completely arbitrary. We assume a frictionless market, meaning that the two assets can be traded in arbitrary amounts, positive and negative, with no costs.

There is arbitrage in this model if $Rs_0 \leq s_{12}$ or $Rs_0 \geq s_{11}$: in these cases borrowing from the bank and investing in stock, or *vice versa*, realizes a riskless profit with positive probability. We therefore suppose that $s_{12} < Rs_0 < s_{11}$.

Suppose that at time 0 we form a portfolio consisting of $\$B$ in the bank and N shares of stock. The value of this portfolio is $B + NS_0$ and its value at time 1 will be either $RB + Ns_{11}$ or $RB + Ns_{12}$. Suppose we choose B, N to satisfy the linear equations

$$\begin{aligned} RB + Ns_{11} &= a_{11} \\ RB + Ns_{12} &= a_{12}, \end{aligned}$$

to which the solution is

$$N = \frac{a_{11} - a_{12}}{s_{11} - s_{12}}, \quad B = \frac{a_{12}s_{11} - a_{11}s_{12}}{R(s_{11} - s_{12})}. \quad (4)$$

With this choice of N, B the value of our portfolio coincides with the option exercise value, whichever way the price moves. We say the portfolio *replicates* the option payoff. By the Law of One Price, the value of the option at time 0 must be equal to the value of the portfolio at

time 0, which is

$$A_0 \equiv B + NS_0 = \frac{1}{R}(a_{12}s_{11} - a_{11}s_{12}) + \frac{a_{11} - a_{12}}{s_{11} - s_{12}}s_0. \quad (5)$$

We have shown that there is a unique arbitrage-free price for the option, obtained by calculating the ‘perfect hedging’ strategy (B, N) . This is the essence of the Black-Scholes argument. However, more can be said. We can rearrange the price formula (5) to read

$$A_0 = \frac{1}{R}(qa_{11} + (1 - q)a_{12}),$$

where $q = (Rs_0 - s_{12})/(s_{11} - s_{12})$. Note that q does not depend on the option contract and that our no-arbitrage assumption $s_{12} < Rs_0 < s_{11}$ is equivalent to the statement that $0 < q < 1$. We can therefore interpret $q, (1 - q)$ as probabilities of an upward and downward move respectively, and rewrite (5) again as

$$A_0 = E_q \left[\frac{1}{R}A_1 \right]. \quad (6)$$

This states that the option value is the expected value, under the probability measure defined by q , of the discounted payoff. This probability measure is called the *risk-neutral* or *equivalent martingale* measure. The latter terminology arises from a further characterization of q , evident from (6). Indeed, if we put $a_{11} = s_{11}, a_{12} = s_{12}$ then $A_0 = S_0$ and we see from (6) that the process $\tilde{S}_k, k = 0, 1$ of discounted asset prices, defined by $\tilde{S}_0 = S_0, \tilde{S}_1 = S_1/R$, is a martingale. q is the unique upward probability such that this is so.

The key thing to realize here is that q is not the actual probability of an upward move. We were very careful not to state what this probability is, but only to say that upward and downward moves both occur with positive probability (and that there is no other sort of move). In technical terms, this means that the measure defined by q is ‘equivalent’ to the actual probability.

To summarize, we have shown:

- The model is arbitrage-free if and only if there is a unique equivalent martingale measure (EMM).
- The EMM—if it exists—is the unique probability measure such that the discounted asset price is a martingale.
- Any contingent claim has a unique value consistent with the absence of arbitrage.
- This value is the initial investment required to replicate the claim by trading in the market.
- The value can be expressed as the expectation of the discounted exercise value, calculated under the EMM.

The question at issue now is whether these five statements continue to hold for other, more realistic, market models. Remarkably, they are all true for the Black-Scholes model—and even for some generalizations of the Black-Scholes model—but it took some years for the full picture to emerge.

6 Black-Scholes and beyond

Viewed from a latter-day perspective, the great paper of Black and Scholes in 1973 (hereafter B&S) comes as a huge breath of fresh air. They clear away all the clutter and get straight to the point, announcing their manifesto in the first two sentences of the abstract:

If options are correctly priced in the market, it should not be possible to make sure profits by creating portfolios of long and short positions in options and their underlying stocks. Using this principle, a theoretical valuation formula for options is derived.

They make what are now the standard assumptions of frictionless markets and adopt Samuelson's geometric Brownian motion price model (3) (no dividends are paid). They only consider European options. Then the value of a call option with strike K and exercise time T should just depend on the current time t and underlying price S_t , i.e. it should be equal to $w(S_t, t)$ for some function $w(x, t)$. They recognize immediately that the way to form riskless portfolios is 'delta hedging'. Suppose $S_t = x$ and we form a portfolio consisting of one unit of the underlying asset and $-(1/w_1(x, t))$ units of the option, where $w_1 = \partial w / \partial x$. The value of the portfolio is $p = x - w/w_1$ and the change in value in a short time Δt is $\Delta p = \Delta x - \Delta w/w_1$. Expanding w by the Itô formula gives

$$\Delta w = w_1 \Delta x + \frac{1}{2} w_{11} \sigma^2 x^2 \Delta t + w_2 \Delta t$$

and hence

$$\Delta p = -\frac{1}{w_1} \left(\frac{1}{2} w_{11} \sigma^2 x^2 + w_2 \right) \Delta t. \quad (7)$$

Since this return is certain, it must coincide with the return on the riskless asset, i.e.

$$\Delta p = rp \Delta t = r \left(x - \frac{w}{w_1} \right) \Delta t. \quad (8)$$

Equating the right-hand sides of (7) and (8) and cancelling Δt gives the Black-Scholes PDE

$$w_2 + w_1 r x + \frac{1}{2} w_{11} \sigma^2 x^2 - r w = 0, \quad (9)$$

and of course we know the value at $t = T$, namely $w(T, x) = \max(x - K, 0)$. B&S introduce a change of variables which reduces (9) to the standard heat equation with constant coefficients, which they can solve in integral form, giving the famous formula

$$w(x, t) = xN(d_1) - Ke^{-r(T-t)}N(d_2), \quad (10)$$

where N is the cumulative normal distribution function and

$$d_1 = \frac{\log(x/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Curiously, they obtain the corresponding value $u(x, t)$ for a put option by noting that the difference $w - u$ satisfies (9) with the boundary condition $w(x, T) - u(x, T) = x - K$, to which the solution is easily seen to be

$$w(x, t) - u(x, t) = x - e^{-r(T-t)}K. \quad (11)$$

In fact this equality, known as ‘put-call parity’, is much more fundamental than the B&S formula itself. It follows directly from an arbitrage argument¹⁰ and is not connected with the B&S price model.

Regrettably, Fischer Black did not live to collect the economics Nobel Prize for this extraordinary achievement. The 1997 prize¹¹ was awarded to Myron Scholes and Robert Merton ‘for a new method to determine the value of derivatives’. Merton’s own paper on option pricing (Merton, 1973), contains a cornucopia of good ideas most of which have now found their way into the standard theory. He introduces a more formal concept of trading strategies and self-financing portfolios. Perhaps most significantly, he allows for stochastic interest rates, introducing as a second asset the zero coupon bond $p(t, T)$ (the value at time t of \$1 delivered at time $T \geq t$), which is assumed to satisfy

$$dp(t, T) = \mu(t, T)p(t, T)dt + \delta(t, T)p(t, T)dz_T(t),$$

where μ, δ are deterministic functions and $z_T(t)$ is another Brownian motion. Thus p , as well as the asset price S , has log-normal distribution. The option price $w(x, p, t)$ must now depend on the current prices $x = S(t)$ and $p = p(t, T)$ of both assets. Merton chooses a ‘delta-hedging’ strategy in both assets to obtain a portfolio that is ‘locally riskless’ and obtains a PDE in three variables for the option price w . He now observes that w has the homogeneity property that $w(\lambda x, \lambda p, t) = \lambda w(x, p, t)$ for $\lambda > 0$. Taking $\lambda = 1/p$ we obtain

$$w(x, p, t) = pw\left(\frac{x}{p}, 1, t\right),$$

expressing the option value in terms of a two-parameter function $h(y, t) \equiv w(y, 1, t)$ which is shown to satisfy the B&S PDE with a modified volatility term. This of course reduces to standard B&S when the zero-coupon bond has no volatility, i.e., $\delta(t, T) = 0$.

The significance of this development, apart from the introduction of interest-rate volatility, is the realization that prices are ratios, i.e. should be expressed in units of some *numéraire asset* (in this case, the zero-coupon bond p). Interest-rate modelling soon took off as a subject in its own right, and the numéraire idea is now commonplace in the option pricing literature. Later studies by Jamshidian (1997) and others have systematically exploited homogeneity properties of price functions. The germs of all these ideas are in Merton’s paper.

The next big step forward was taken by Cox and Ross (1976). They give a proof of Black-Scholes which is pretty close to what is found nowadays in dozens of textbooks. Interestingly, though, they extend the argument to cover a limited class of jump-diffusion processes, that is price processes in which the input ‘noise’ is Brownian motion plus a Poisson-like jump process. The price process is therefore discontinuous, but Cox and Ross show that nevertheless perfect hedging can be achieved in certain circumstances. They are also able to handle assets paying a dividend yield, a topic with which, as we pointed out above, Black and Scholes had had some

¹⁰As Bachelier pointed out on page ??, being long a call option and short a put option is equivalent to holding a forward contract; the values of these two equivalent positions are respectively the left and right hand sides of (11).

¹¹Its full name is The Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel.

difficulty. The main significance of the Cox-Ross paper, however, lies in the introduction of ‘risk-neutral valuation’. They note that valuation is preference-independent. Therefore it makes no difference what investor preferences are assumed. If we assume that investors are risk-neutral, then economic equilibrium requires that the expected returns of all assets be equal to the riskless rate. Option prices can be expressed as discounted expectations in this ‘risk-neutral world’. This key idea, specialized to the single-period binomial model, gives the pricing formula (6) that we derived above.

The binomial model itself was introduced by Cox and Ross again, together with their collaborator Mark Rubinstein, in 1979. Again from today’s perspective, there is a quite noticeable shift in style in this paper away from ‘economics’ and towards what is now known as ‘financial engineering’: an accent on computable models and effective practical techniques. And it is certainly true that the binomial tree, introduced in this paper, soon became, and to some extent remains today, one of the widely-used workhorses of the trading environment.

The binomial tree is a discrete-time process consisting of n independent steps of the sort analysed above. Starting at any node, the price moves up by a factor $u > 1$ or down by a factor $d < 1$. By taking $d = 1/u$ and normalizing to an initial price $S_0 = 1$ we get a ‘recombining tree’ with the prices at the nodes as shown in the figure. As in the 1-period model described earlier, we assume the per-period riskless interest rate is r . The condition for absence of arbitrage is then $d < R < u$ as before, where $R = 1 + r$. Define $h = \log u$, so that $u = e^h$, $d = e^{-h}$, and let Z_1, Z_2, \dots be i.i.d. random variables with $P[Z_k = 1] = P[Z_k = -1] = q = (R - d)/(u - d)$. Then the price process can be expressed as

$$S_k = S_0 \exp \left(h \sum_{i=1}^k Z_i \right),$$

and P is the risk-neutral measure in that the discounted price process $\tilde{S}_k = S_k/R^k$ is a martingale. S_k is a ‘multiplicative random walk’. If we have an option whose exercise value at time n is $f(S_n)$ then its unique arbitrage-free value at time 0 is the discounted expectation $w(S_0, 0) = R^{-n}E[f(S_n)]$. This is proved by constructing a hedging portfolio by a direct extension of the 1-period case. Also included in Cox et al. (1979) is the scaling argument that leads to the use of the binomial tree as an approximation to Black-Scholes: the multiplicative random walk, suitably scaled, approximates geometric Brownian motion, and the binomial tree is in fact providing a simple algorithm for solving the PDE (9). This is the reason for the enormous impact the binomial model has had for practical option trading.

Another reason for the impact of the binomial model is that, as already pointed out in Cox et al. (1979), we can also use it to price American options. The idea is as follows. The right-hand side of (??) gives the ‘continuation value’ of the option as seen at time k . In the European case, this *is* the value since one has no choice but to wait until the final time n before exercising. But in the American case, the holder could exercise *now*, at time k , getting the exercise value $f(s)$. Rationally, he will do this if $f(s)$ is greater than the continuation value, leading to the following

relationship between the American values $w_a(\cdot, k)$ and $w_a(\cdot, k + 1)$:

$$w_a(s, k) = \max \left\{ f(s), \frac{1}{R} \{ q w_a(us, k + 1) + (1 - q) w_a(ds, k + 1) \} \right\}. \quad (12)$$

On the other hand, at time n the option *must* be exercised, giving $w_a(n, s) = f(s)$. We can therefore generate the American price by backwards recursion using (12) together with the same time- n terminal exercise value as in the European case. From (??) it is clear that $w_a(s, k) \geq f(s)$ and that it is optimal to exercise at time s if $w_a(s, k) = f(s)$. Regarding the binomial model as an approximation to the geometric Brownian motion model, this algorithm, when applied to the put option $f(s) = \max(K - s, 0)$, can be seen as a way of computing the solution to the free boundary problem introduced by Samuelson (1965).

All of this is contained in the Cox, Ross and Rubinstein paper, and all of it is absolutely correct, but the argument given in the paper is far from complete. It is not at all obvious why the investor should compare the immediate exercise value with the *risk-neutral* expectation of the continuation value. Some separate argument is required to show that this procedure leads to a unique arbitrage-free price. Not until the work of Bensoussan (1984) and Karatzas (1988), some twenty years after McKean's original work, was the connection between free boundary problems and a unique arbitrage free value for the American option established in a mathematically watertight way.

Cox and Ross (1976) had arrived at the idea of 'risk-neutral valuation' by observing that the Black-Scholes formula (10) does not involve the growth rate α of the stock (3), and therefore the value is the same whatever the growth rate. If investors are 'risk-neutral' they will assume that the growth rates of all assets, risky or not, is equal to the riskless rate r . But if we take $\alpha = r$ in the Black-Scholes model then the option value given by (10) is equal to the discounted expectation $\mathbb{E}[e^{-r(T-t)} \max(S_T - K, 0)]$. Cox and Ross did not, however, make the connection with equivalent martingale measures. This final piece of the jigsaw puzzle was inserted by Harrison and Kreps (1979).

Two probability measures \mathbb{P}, \mathbb{Q} on the same probability space (Ω, \mathcal{F}) are *equivalent* if they have the same null sets: for any measurable set A we have $\mathbb{P}[A] = 0$ if and only if $\mathbb{Q}[A] = 0$. When (Ω, \mathcal{F}) is the standard probability space for Brownian motion and \mathbb{P} is Wiener measure, there is a striking characterization of equivalent measures, uncovered by Girsanov (1960). This is that, for Brownian motion $w(t)$, *change of measure is change of drift*, i.e. under any equivalent measure \mathbb{Q} there is a 'drift' process $\phi(t)$ such that $\tilde{w}(t)$ defined by

$$\tilde{w}(t) = w(t) - \int_0^t \phi(s) ds$$

is a Brownian motion under measure \mathbb{Q} . More intuitively, we can say that, under \mathbb{Q} , $dw(t) = d\tilde{w}(t) + \phi(t)dt$, so $w(t)$ is 'Brownian motion plus drift'. If we look at the geometric Brownian motion (3) we see that by taking $\phi(t) = (r - \alpha)/\sigma$ we obtain

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{w}(t),$$

so that under \mathbb{Q} the price process $S(t)$ is geometric Brownian motion with the risk-neutral growth rate r . We call \mathbb{Q} the equivalent martingale measure (EMM) because under \mathbb{Q} the

discounted price process $e^{-rt}S(t)$ is a martingale. The Black-Scholes formula is expressed as $\mathbb{E}_Q[e^{-r(T-t)} \max(S_T - K, 0)]$, where \mathbb{E}_Q denotes expectation under measure \mathbb{Q} - exactly the formula (6) obtained in the binomial model. Thus instead of appealing to the economic concept of risk-neutrality we can simply state that the option value is the discounted expectation of the exercise value under the unique EMM.

The above idea is contained in the Harrison and Kreps (1979) paper, but they start from a much more abstract point of view, initially in a single-period setting similar to that of the single-period binomial model discussed above, but allowing for a very general set of random outcomes at the end of the period. Agents in the market have available ‘consumption bundles’ $(r, x) \in \mathbb{R} \times \mathbb{X}$ where \mathbb{X} is a linear space of random variables. Thus an agent will consume a fixed amount r at time 0 and a random amount $X(\omega)$ at time 1, for some $X \in \mathbb{X}$. A price system is a pair (\mathbb{M}, π) where \mathbb{M} is a subspace of \mathbb{X} and π is a linear functional on \mathbb{M} . The price of a bundle $(r, m) \in \mathbb{R} \times \mathbb{M}$ is then $r + \pi(m)$. The price system is *viable* if there is an optimal net trade (a bundle such that $r + \pi(m) = 0$) with respect to some preference ordering. The pricing problem is to extend π to all of \mathbb{X} in such a way that the extended market is still viable. A specific claim $X \in \mathbb{X}$ is *priced by arbitrage* if there is a unique price p for X which is consistent with (\mathbb{M}, π) . The main general results are that (\mathbb{M}, π) is viable if and only if there is an extension of π to all of \mathbb{X} which is continuous and strictly positive; a claim X is priced by arbitrage if and only if it has the same value for all such extensions.

Moving to a multi-period or continuous-time setting where the time interval is $[0, T]$, Harrison and Kreps consider a vector process $S(t)$ of asset prices, and restrict their investors to simple self-financing trading strategies.¹² Starting at time 0 with a certain endowment and applying a trading strategy leads to a random outcome X at time T , to which the single-period theory can be applied. The idea is that (\mathbb{M}, π) represents the market of traded assets, while the bigger set \mathbb{X} includes possibly non-marketed contingent claims. The main result is essentially that viability is equivalent to existence of an equivalent martingale measure (EMM) Q . There is a 1-1 correspondence between EMMs Q and pricing functionals ψ , given by $\psi(X) = E_Q X$ and $QA = \psi(\mathbf{1}_A)$. The value of X is determined by arbitrage if $E_Q X$ is the same for every EMM Q . Harrison and Kreps apply this result to finite models similar to the binomial model, and to the Black-Scholes set-up where they bring in the Girsanov theorem to effect the measure change. The paper therefore gives the EMM idea in a variety of settings, but not in any great generality because of the small class of trading strategies to which they restrict themselves.

Harrison continued the quest with co-author Pliska in a further paper in 1981, in which the connection with modern stochastic analysis—the *théorie générale des processus*—was firmly established. This time asset prices are semimartingales and trading strategies are the integrands of the *théorie générale*. They do not directly attack the question of necessary and sufficient conditions for the absence of arbitrage but, rather, assume the existence of an EMM (which is a sufficient condition). They show that the price π of an attainable claim X is always given by $\pi = \mathbb{E}_Q[e^{-rT} X]$. They recognize that a market is ‘complete’ (all claims are attainable) if the vector process of asset prices has the martingale representation property, so that in particular

¹²See the introductory chapter for a description.

the Black-Scholes model is complete, and other models will be complete only if they satisfy general conditions given by Jacod and Yor (1977). This paper has turned ‘financial economics’ into ‘mathematical finance’. All questions are posed in purely mathematical terms, and no economic principles that do not have a precise mathematical statement appear.

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